Newman Janis Algorithm

1 Calculations

The standard kerr metric is given by

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4Mar\sin^2\theta}{\rho^2} dtd\phi + \frac{\Sigma}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

where,
$\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2$ and $\Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$

$$g_{\mu\nu} = \begin{vmatrix} -\left(1-\frac{2Mr}{\rho^2}\right) & 0 & 0 & -\frac{2Mar\sin^2\theta}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{2Mar}{\sin^2\theta}\rho^2 & 0 & 0 & \frac{\Sigma}{\rho^2}\sin^2\theta \end{vmatrix}$$

and the Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

Switching to Eddington-Finkelstein coordinate with $x^{0'} = t + r$, we have

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

And the metric looks like:

$$g_{\mu\nu} = \begin{vmatrix} -\left(1-\frac{2M}{r}\right) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{vmatrix} \quad \text{and} \quad g^{\mu\nu} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 1-\frac{2M}{r} & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{vmatrix}$$

Based on the choice of normalization $l \cdot n = 1$ and $m \cdot \bar{m} = -1$,we have:

$$g^{\mu\nu} = l^\mu n^\nu + n^\mu l^\nu - m^\mu \bar{m}^\nu - \bar{m}^\mu m^\nu$$

In advanced Eddington-Finkelstein coordinate, the null geodesic is directed towards ∂_v, thus let us take that as l^μ and assuming $m_v = m_r = 0$. Thus, the null tetrad can be calculated from the metric tensor in this coordinate and is given by:

$$l^\mu = (0, 1, 0, 0) \tag{1}$$

$$n^\mu = \left(1, \frac{1}{2}\left(1 - \frac{2M}{r}\right), 0, 0\right) \tag{2}$$

$$m^\mu = \frac{1}{\sqrt{2}r}\left(0, 0, 1, \frac{\iota}{\sin\theta}\right) \tag{3}$$

$$\bar{m}^\mu = \frac{1}{\sqrt{2}r}\left(0, 0, 1, -\frac{\iota}{\sin\theta}\right) \tag{4}$$

Under the Newman-Janis transformation, $r' \to r + \imath a\cos\theta$ and $v' \to v - \imath a\cos\theta$, we have:

$$\begin{aligned} l'^a &= \frac{\partial x'^a}{\partial x^a} l^a = \frac{\partial x'^a}{\partial r} l^r \\ &= \delta^a_r \end{aligned}$$

$$\begin{aligned} n'^a &= \frac{\partial x'^a}{\partial x^a} n^a = \frac{\partial x'^a}{\partial v} n^v + \frac{\partial x'^a}{\partial r} n^r \\ &= \delta^a_v - \frac{f(r)}{2}\delta^a_r \\ &= \delta^a_v - \frac{1}{2}\left(1 - M\frac{1}{r} - M\frac{1}{\bar{r}}\right)\delta^a_r \\ &= \delta^a_v - \frac{1}{2}\left(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\right)\delta^a_r \end{aligned}$$

The last two of the tetrads can be calculated as follows:

$$\frac{\partial r'}{\partial\theta} = \frac{\partial}{\partial\theta}(r + \imath a\cos\theta) = -\imath a\sin\theta \tag{5}$$

$$\frac{\partial v'}{\partial\theta} = \frac{\partial}{\partial\theta}(v - \imath a\cos\theta) = \imath a\sin\theta \tag{6}$$

Now, moving onto the tetrads

$$m^a = \frac{1}{\sqrt{2}r}\left(\delta^a_\theta + \frac{\imath}{\sin\theta}\delta^a_\phi\right) \qquad \text{and} \qquad \bar{m}^a = \frac{1}{\sqrt{2}\bar{r}}\left(\delta^a_\theta - \frac{\imath}{\sin\theta}\delta^a_\phi\right)$$

Transformed tetrads are given as:

$$\begin{aligned}
m'^a &= \frac{\partial x'^a}{\partial x^a} m^a = \frac{\partial x'^a}{\partial x^\theta} m^\theta + \frac{\partial x'^a}{\partial \phi} m^\phi \\
&= \left(\frac{\partial x'^a}{\partial r'} \frac{\partial r'}{\partial \theta} + \frac{\partial x'^a}{\partial v'} \frac{\partial v'}{\partial \theta} + \frac{\partial x'^a}{\partial \theta'} \frac{\partial \theta'}{\partial \theta} \right) m^\theta + \frac{\partial x'^a}{\partial \phi} m^\phi \\
&= (-\imath a \sin\theta \delta^a_{r'} + \imath a \sin\theta \delta^a_{v'} + \delta^a_\theta)\, m^\theta + \frac{\partial x'^a}{\partial \phi} m^\phi \\
&= \frac{1}{\sqrt{2} r} \left(\delta^a_\theta + (\delta^a_{v'} - \delta^a_{r'}) \imath a \sin\theta + \frac{1}{\sin\theta} \delta^a_\phi \right) \qquad (\text{using } m^\theta = \tfrac{1}{\sqrt{2}r} \delta^\theta_\theta) \\
&= \frac{1}{\sqrt{2}(r' - \imath a \cos\theta)} \left(\delta^a_\theta + (\delta^a_{v'} - \delta^a_{r'}) \imath a \sin\theta + \frac{1}{\sin\theta} \delta^a_\phi \right)
\end{aligned}$$

Now, $\bar{m}^a$

$$\begin{aligned}
\bar{m}'^a &= \frac{\partial x'^a}{\partial x^a} \bar{m}^a = \frac{\partial x'^a}{\partial x^\theta} \bar{m}^\theta + \frac{\partial x'^a}{\partial \phi} \bar{m}^\phi \\
&= \left(\frac{\partial x'^a}{\partial \bar{r}'} \frac{\partial \bar{r}'}{\partial \theta} + \frac{\partial x'^a}{\partial \bar{v}'} \frac{\partial \bar{v}'}{\partial \theta} + \frac{\partial x'^a}{\partial \theta'} \frac{\partial \theta'}{\partial \theta} \right) \bar{m}^\theta + \frac{\partial x'^a}{\partial \phi} \bar{m}^\phi \\
&= (\imath a \sin\theta \delta^a_{r'} - \imath a \sin\theta \delta^a_{v} + \delta^a_\theta)\, \bar{m}^\theta + \frac{\partial x'^a}{\partial \phi} \bar{m}^\phi \\
&= \frac{1}{\sqrt{2} \bar{r}} \left(\delta^a_\theta - (\delta^a_{v} - \delta^a_{r'}) \imath a \sin\theta - \frac{1}{\sin\theta} \delta^a_\phi \right) \qquad (\text{using } \bar{m}^\theta = \tfrac{1}{\sqrt{2}\bar{r}} \delta^\theta_\theta) \\
&= \frac{1}{\sqrt{2}(r' + \imath a \cos\theta)} \left(\delta^a_\theta - (\delta^a_{v} - \delta^a_{r'}) \imath a \sin\theta - \frac{1}{\sin\theta} \delta^a_\phi \right)
\end{aligned}$$

Thus the relevant metric tensor is given as:

$$\begin{aligned}
g^{vv} &= l^v n^v + n^v l^v - m^v \bar{m}^v - \bar{m}^v m^v \\
&= 0 \times \delta^v_v + \delta^v_v \times 0 - \frac{\imath a \sin\theta}{\sqrt{2}(r' - \imath a \cos\theta)} \times \frac{-\imath a \sin\theta}{\sqrt{2}(r' + \imath a \cos\theta)} \\
&\qquad - \frac{-\imath a \sin\theta}{\sqrt{2}(r' + \imath a \cos\theta)} \times \frac{\imath a \sin\theta}{\sqrt{2}(r' - \imath a \cos\theta)} \\
&= -\frac{a^2 \sin^2\theta}{r'^2 + a^2 \cos^2\theta}
\end{aligned}$$

$$\begin{aligned}
g^{rr} &= l^r n^r + n^r l^r - m^r \bar{m}^r - \bar{m}^r m^r \\
&= 1 \times -\frac{f(r)}{2} - \frac{f(r)}{2} \times 1 - \frac{-\imath a \sin\theta}{\sqrt{2}(r' - \imath a \cos\theta)} \times \frac{\imath a \sin\theta}{\sqrt{2}(r' + \imath a \cos\theta)} \\
&\qquad - \frac{\imath a \sin\theta}{\sqrt{2}(r' + \imath a \cos\theta)} \times \frac{-\imath a \sin\theta}{\sqrt{2}(r' - \imath a \cos\theta)}
\end{aligned}$$

$$= -\left(1 - \frac{2Mr}{r'^2 + a^2\cos^2\theta}\right) - \frac{a^2\sin^2\theta}{r'^2 + a^2\cos^2\theta}$$
$$= -\left(\frac{r^2 + a^2\cos^2\theta - 2Mr + a^2\sin^2\theta}{r'^2 + a^2\cos^2\theta}\right) = -\left(\frac{r^2 - 2Mr + a^2}{r'^2 + a^2\cos^2\theta}\right)$$

$$\begin{aligned}
g^{\theta\theta} &= l^\theta n^\theta + n^\theta l^\theta - m^\theta \bar{m}^\theta - \bar{m}^\theta m^\theta \\
&= 0\times 0 + 0\times 0 - \frac{1}{\sqrt{2}(r' - \imath a\cos\theta)}\frac{1}{\sqrt{2}(r' + \imath a\cos\theta)} \\
&\qquad - \frac{1}{\sqrt{2}(r' + \imath a\cos\theta)}\frac{1}{\sqrt{2}(r' - \imath a\cos\theta)} \\
&= -\frac{1}{(r'^2 + a^2\cos^2\theta)}
\end{aligned}$$

$$\begin{aligned}
g^{\phi\phi} &= l^\phi n^\phi + n^\phi l^\phi - m^\phi \bar{m}^\phi - \bar{m}^\phi m^\phi \\
&= 0\times 0 + 0\times 0 - \frac{\imath}{\sqrt{2}(r' - \imath a\cos\theta)\sin\theta}\frac{-\imath}{\sqrt{2}(r' + \imath a\cos\theta)\sin\theta} \\
&\qquad - \frac{-\imath}{\sqrt{2}(r' + \imath a\cos\theta)\sin\theta}\frac{\imath}{\sqrt{2}(r' - \imath a\cos\theta)\sin\theta} \\
&= -\frac{1}{(r^2 + a^2\cos^2\theta)\sin^2\theta}
\end{aligned}$$

$$\begin{aligned}
g^{vr} &= l^v n^r + n^v l^r - m^v \bar{m}^r - \bar{m}^v m^r \\
&= 0\times\frac{-f(r)}{2} + 1\times 1 - \frac{\imath a\sin\theta}{\sqrt{2}(r' - \imath a\cos\theta)}\frac{\imath a\sin\theta}{\sqrt{2}(r' - \imath a\cos\theta)} \\
&\qquad - \frac{-\imath a\sin\theta}{\sqrt{2}(r' - \imath a\cos\theta)}\frac{-\imath a\sin\theta}{\sqrt{2}(r' - \imath a\cos\theta)} \\
&= 1 + \frac{a^2\sin^2\theta}{r^2 + a^2\cos^2\theta}
\end{aligned}$$

$$\begin{aligned}
g^{rv} &= l^r n^v + n^r l^v - m^r \bar{m}^v - \bar{m}^r m^v \\
&= 1\times 1 + 1\times 0 - \frac{-\imath a\sin\theta}{\sqrt{2}(r' - \imath a\cos\theta)}\frac{-\imath a\sin\theta}{\sqrt{2}(r' - \imath a\cos\theta)} \\
&\qquad - \frac{\imath a\sin\theta}{\sqrt{2}(r' - \imath a\cos\theta)}\frac{\imath a\sin\theta}{\sqrt{2}(r' - \imath a\cos\theta)} \\
&= 1 + \frac{a^2\sin^2\theta}{r^2 + a^2\cos^2\theta}
\end{aligned}$$

$$\begin{aligned}
g^{v\theta} &= l^v n^\theta + n^v l^\theta - m^v \bar{m}^\theta - \bar{m}^v m^\theta \\
&= 0\times 0 + 0\times 0 - \frac{\imath a\sin\theta}{\sqrt{2}(r - \imath a\cos\theta)}\times\frac{1}{\sqrt{2}(r' + \imath a\cos\theta)} \\
&\qquad - \frac{-\imath a\sin\theta}{\sqrt{2}(r + \imath a\cos\theta)}\times\frac{1}{\sqrt{2}(r' - \imath a\cos\theta)} \\
&= 0
\end{aligned}$$

$$g^{v\phi} = l^v n^\phi + n^v l^\phi - m^v \bar{m}^\phi - \bar{m}^v m^\phi$$

$$
\begin{aligned}
&= 0 \times 0 + 0 \times 0 - \frac{\imath a \sin\theta}{\sqrt{2}(r - \imath a \cos\theta)} \frac{-\imath}{\sqrt{2}(r - \imath a \cos\theta)\sin\theta} \\
&\qquad\qquad - \frac{-\imath a \sin\theta}{\sqrt{2}(r + \imath a \cos\theta)} \times \frac{\imath}{\sqrt{2}(r + \imath a \cos\theta)\sin\theta} \\
&= -\frac{a}{r^2 + a^2\cos^2\theta}
\end{aligned}
$$

$$
\begin{aligned}
g_{r\phi} &= l^r n^\phi + n^r l^\phi - m^r \bar{m}^\phi - \bar{m}^r m^\phi \\
&= 0 \times 0 + 0 \times 0 - \frac{-\imath a \sin\theta}{\sqrt{2}(r - \imath a \cos\theta)} \frac{-\imath}{\sqrt{2}(r - \imath a \cos\theta)\sin\theta} \\
&\qquad\qquad - \frac{\imath a \sin\theta}{\sqrt{2}(r + \imath a \cos\theta)} \times \frac{\imath}{\sqrt{2}(r + \imath a \cos\theta)\sin\theta} \\
&= \frac{a}{r^2 + a^2\cos^2\theta}
\end{aligned}
$$

Therefore the metric tensor takes the following form:

$$
g^{\mu\nu} = \begin{vmatrix}
-\frac{a^2\sin^2\theta}{r'^2+a^2\cos^2\theta} & 1 + \frac{a^2\sin^2\theta}{r^2+a^2\cos^2\theta} & 0 & -\frac{a}{r^2+a^2\cos^2\theta} \\
1 + \frac{a^2\sin^2\theta}{r^2+a^2\cos^2\theta} & -\frac{r^2-2Mr+a^2}{r'^2+a^2\cos^2\theta} & 0 & \frac{a}{r^2+a^2\cos^2\theta} \\
0 & 0 & -\frac{1}{(r'^2+a^2\cos^2\theta)} & 0 \\
-\frac{a}{r^2+a^2\cos^2\theta} & \frac{a}{r^2+a^2\cos^2\theta} & 0 & -\frac{1}{(r^2+a^2\cos^2\theta)\sin^2\theta}
\end{vmatrix}
$$

The inverse metric is given by

$$
g^{\mu\nu} = \begin{vmatrix}
1 - \frac{2Mr}{r^2+a^2\cos^2\theta} & 1 & 0 & \frac{2Mar\sin^2\theta}{r^2+a^2\cos^2\theta} \\
1 & 0 & 0 & -a\sin^2\theta \\
0 & 0 & -\frac{1}{r^2+a^2\cos^2\theta} & 0 \\
\frac{2Mar\sin^2\theta}{r^2+a^2\cos^2\theta} & -a\sin^2\theta & 0 & -\left[(r^2+a^2) + \frac{2Mar\sin^2\theta}{r^2+a^2\cos^2\theta}\right]\sin^2\theta
\end{vmatrix}
$$

This result matches with Chandrasekhar's Mathematical Theory of Black Hole, pg 307. The Newman Janis Algorithm is Einstein Field Invariant, it doesn't break the diffeomorphism invariance.